

Coding for Network-Coded Slotted ALOHA

Shenghao Yang*, Yi Chen[†], Soung Chang Liew[‡] and Lizhao You[‡]

*Institute of Network Coding, [†]Information Engineering, The Chinese University of Hong Kong, Hong Kong, China

[‡]Electronic Engineering, City University of Hong Kong, Hong Kong, China

Abstract—Slotted ALOHA can benefit from physical-layer network coding (PNC) by decoding one or multiple linear combinations of the packets simultaneously transmitted in a timeslot, forming a system of linear equations. Different systems of linear equations are recovered in different timeslots. A message decoder then recovers the original packets of all the users by jointly solving multiple systems of linear equations obtained over different timeslots. We propose the batched BP decoding algorithm that combines belief propagation (BP) and local Gaussian elimination. Compared with pure Gaussian elimination decoding, our algorithm reduces the decoding complexity from cubic to linear function of the number of users. Compared with the ordinary BP decoding algorithm for low-density generator-matrix codes, our algorithm has better performance and the same order of computational complexity. We analyze the performance of the batched BP decoding algorithm by generalizing the tree-based approach and provide an approach to optimize the system performance.

I. INTRODUCTION

In a wireless multiple-access network operated with the slotted ALOHA access protocol, a number of users transmit messages to a sink node through a common wireless medium. Time is divided into discrete slots and all transmissions start at the beginning of a timeslot. Collisions/interferences occur when more than one user transmits in the same timeslot [1]. Successive interference cancellation (SIC) was proposed for slotted ALOHA to resolve collisions so that signals contained in collisions can be leveraged to increase throughput [2]. In this approach, a number of timeslots are grouped together as a frame. Each user aims to deliver at most one packet per frame, but it can transmit copies of the same packet in different timeslots of the frame.

To see the essence, suppose we have two users and the first user transmits two copies of packet v_1 in timeslots 1 and 2 respectively, and the second user transmits one copy of packet v_2 in timeslot 2. Since no collision occurs, v_1 can be correctly decoded by the sink node in timeslot 1. A collision occurs in timeslot 2. In SIC, the sink node can use v_1 decoded from timeslot 1 to cancel the interference in timeslot 2. This approach can be applied iteratively to cancel more interference, in a manner similar to the belief propagation (BP) decoding of LT codes over erasure channels. Slotted ALOHA

with SIC has been extensively studied based on the AND-OR-tree analysis, and optimal designs have been obtained [3]–[7].

Physical-layer network coding (PNC) [8] (also known as *compute-and-forward* [9]) is recently applied to wireless multiple-access network to improve the throughput [10]–[12]. Such multiple-access schemes, called *network-coded multiple access (NCMA)*, employ both PNC and multiuser decoders at the physical layer to decode one or multiple linear combinations of the packets simultaneously transmitted in a timeslot. Specifically, Lu, You and Liew [10] demonstrated by a prototype that a PNC decoder may sometimes successfully recover linear combinations of the packets when the traditional multiuser decoder (MUD) [13] that does not make use of PNC fails. In the existing works on PNC (or compute-and-forward), the decoding of the XOR of the packets of two users has been extensively investigated [14], [15] (see also the overview [16]). The decoding of multiple linear combinations over a larger alphabet has been studied in [9], [17], [18].

In this paper, we consider slotted ALOHA employing PNC (and MUD) at the physical layer, called *network-coded slotted ALOHA (NCSA)*. We assume that the physical-layer decoder at the sink node can reliably recover one or multiple linear combinations of the packets transmitted simultaneously in one timeslot. Our work in this paper does not depend on a specific PNC scheme. Specifically, we consider a K -user NCSA system, where each user has one input packet to be delivered over a frame of timeslots. A packet is the smallest transmission unit, which cannot be further separated into multiple smaller transmission units. But it is allowed to send multiple copies of a packet in different timeslots. The number of copies, called the *degree*, is independently sampled from a *degree distribution*. The linear equations decoded by the physical layer in a timeslot form a system of linear equations. Different systems of linear equations are recovered in different timeslots. To recover the input packets of users, a message decoder is then required to jointly solve these systems of linear equations obtained over different timeslots. Though Gaussian elimination can be applied to solve the input packets, the computational complexity is $O(K^3 + K^2T)$ finite-field operations, where T the number of field symbols in a packet. In this paper, we study the design of NCSA employing an efficient message decoding algorithm.

With the possibility of decoding more than one linear combination of packets in a timeslot, the coding problem induced by NCSA becomes different from that of slotted ALOHA with SIC. We will show by an example that the ordinary

This work was partially supported by the National Natural Science Foundation of China under Grant 61471215. This work was partially supported by the General Research Funds (Project No. 414812) and the AoE grant (Project No. AoE/E-02/08), established under the University Grant Committee of the Hong Kong Special Administrative Region, China.

BP decoding algorithm of LT codes over erasure channels is not optimal for NCSA. We instead propose a *batched BP decoding algorithm* for NCSA, where Gaussian elimination is applied locally to solve the linear system associated with each timeslot, and BP is applied between the linear systems obtained over different timeslots. The computational complexity of our algorithm is $O(KT)$ finite-field operations, which is of the same order as the ordinary BP decoding algorithm. We analyze the asymptotic performance of the batched BP decoding algorithm when K is large by generalizing the tree-based approach in [19]. We provide an approach to optimize the degree distribution based on our analytical results.

Though the batched BP decoding is similar to the one proposed for NCMA [20], [21], we cannot apply the analysis therein. In NCMA, we assume that the number of users is fixed but the number of packets to be delivered by each user tends to infinity. In NCSA, each user has only one packet while the number of users can be large.

Similar schemes have been developed for random linear network coding over finite fields without explicitly considering the physical-layer effect, e.g., BATS codes and chunked codes (see [22], [23] and the references therein). Here the technique for NCSA is different from BATS (or chunked) codes in two aspects. First, in BATS codes the degree distribution of batches is the parameter to be optimized, while in NCSA the degree distribution of the input packets (variable nodes) is the parameter to be optimized. Second, the decoding of BATS codes only solves the associated linear system of a batch when it is uniquely solvable (and hence recovers all the input packets involved in a batch), while the decoding of NCSA processes the associated linear system of a batch even when it is not uniquely solvable.

In the remainder of this paper, Section II formally introduces NCSA and presents our main analytical result (Theorem 2). An outline of the proof of the theorem is given in Section III. An example is provided in Section IV to demonstrate the degree distribution optimization and the numerical results.

II. NETWORK-CODED SLOTTED ALOHA

In this section, we introduce the model of network-coded slotted ALOHA (NCSA), the message decoding algorithm and the performance analysis results.

A. Slotted Transmission

Fix a *base field* \mathbb{F}_q with q elements and an integer $m > 0$. Consider a wireless multiple-access network where K source nodes (users) deliver information to a sink node through a common wireless channel. Each user has one input packet for transmission, formulated as a column vector of T symbols in the extension field \mathbb{F}_{q^m} .

All the users are synchronized to a *frame* consisting of n timeslots of the same duration. The transmission of a packet starts at the beginning of a timeslot, and the timeslots are long enough for completing the transmission of a packet. Each user transmits a number of copies of its input packet within the frame. The number of copies transmitted by a user, called

the degree of the packet, is picked independently according to a *degree distribution* $\Lambda = (\Lambda_1, \dots, \Lambda_D)$, where D is the maximum degree. That is, with probability Λ_d , a user transmits d copies of its input packet in d different timeslots chosen uniformly at random in the frame. Let $\bar{\Lambda} = \sum_{i=1}^D i\Lambda_i$, $\Lambda(x) = \sum_i \Lambda_i x^i$ and $\Lambda'(x) = \sum_i i\Lambda_i x^{i-1}$. We also call $\Lambda(x)$ a degree distribution.

Denote by v_i the input packet of the i -th user. Fix a timeslot. Let Θ be the set of indices of the users who transmit a packet in this timeslot. The elements in Θ are ordered by the natural order of integers. We assume that certain PNC scheme is applied, so that the physical-layer decoder of the sink node can decode multiple *output packets*, each being a linear combination of $v_s, s \in \Theta$ with coefficients over the base field \mathbb{F}_q . Suppose that B output packets are decoded (B may vary from timeslot to timeslot). The collection of B linear combinations can be expressed as

$$[u_1, \dots, u_B] = [v_s, s \in \Theta]\mathbf{H}, \quad (1)$$

where \mathbf{H} is a $|\Theta| \times B$ full-column-rank matrix over \mathbb{F}_q , called the *transfer matrix*, and $[v_s, s \in \Theta]$ is the matrix formed by juxtaposing the vectors v_s , where $v_{s'}$ comes before $v_{s''}$ whenever $s' < s''$.

Note that in (1), the algebraic operations are over the field \mathbb{F}_{q^m} . We call the set of packets $\{u_1, \dots, u_B\}$ decoded in a timeslot a *batch*. The cardinality of Θ (the number of users transmitting in a timeslot) is called the *degree of the batch/timeslot*. We call the ratio K/n the *design rate* of NCSA.

Lemma 1. *When $K/n \rightarrow R$ as $K \rightarrow \infty$, the degree of a timeslot converges to the Poisson distribution with parameter $\lambda = R\bar{\Lambda}$ as $K \rightarrow \infty$.*

Proof: This is a special case of Lemma 6 to be proved later in this paper. ■

Denote by \mathcal{H}_d the collection of all the full-column-rank, d -row matrices over \mathbb{F}_q , where we assume that the empty matrix, representing the case that nothing is decoded, is an element of \mathcal{H}_d . For a timeslot of degree d , we suppose that the transfer matrix of the batch is $\mathbf{H} \in \mathcal{H}_d$ with probability $g(\mathbf{H}|d)$. Further, we consider all the users are symmetric so that for any $d \times d$ permutation matrix \mathbf{P} ,

$$g(\mathbf{H}|d) = g(\mathbf{PH}|d). \quad (2)$$

The transfer matrices of all timeslots are independently generated given the degrees of the timeslots. Examples of the distribution g will be given in Section IV.

We say a rate R is *achievable* by the NCSA system if for any $\epsilon > 0$ and all sufficiently large n , at least $n(R - \epsilon)$ input packets are decoded correctly from the receptions of the n timeslots with probability at least $1 - \epsilon$.

B. Belief Propagation Decoding

For multiple access described above, the goal of the sink node is to decode as many input packets as possible during a frame. From the output packets of the n timeslots decoded by

the physical layer, the original input packets can be recovered by solving the linear equations (1) of all the timeslots jointly. Gaussian elimination has a complexity $O(K^3 + K^2T)$ finite-field operations when $n = O(K)$, which makes the decoding less efficient when K is large.

The output packets of all the timeslots collectively can be regarded as a low-density generator matrix (LDGM) code. Similar to decoding an LT code, which is also a LDGM code, we can apply the (ordinary) BP algorithm to decode the output packets. In each step of the BP decoding algorithm, an output packet of degree one is found, the corresponding input packet is decoded, and the decoded input packet is substituted into the other output packets in which it is involved. The decoding stops when there are no more output packets of degree one. However, as we will show in the next example, the ordinary BP decoding cannot decode some types of batches efficiently. We can actually do better than the ordinary BP decoding with little increase of decoding complexity by exploiting the batch structure of the output packets.

For example, consider a batch of two packets u_1 and u_2 formed by

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (3)$$

Suppose that we use the ordinary BP decoding algorithm, and when the BP decoding stops, v_1 is recovered by processing other batches, but v_2, v_3 and v_4 are not recovered. However, if we allow the decoder to solve the linear system (3), we can further recover $v_2 = u_2 - u_1 + v_1$. The example shows that the BP decoding performance can be improved if the linear system associated with a timeslot can be solved locally.

Motivated by the above example, we propose the *batched BP decoder* for the output packets of the physical layer of NCSA. The decoder includes multiple iterations. In the i -th iteration of the decoding, $i = 1, 2, \dots$ all the batches are processed individually by the following algorithm: Consider a batch given in (1). Let $S \subset \Theta$ be the set of indices r such that v_r is decoded in the previous iterations. When $i = 0$, $S = \emptyset$. Let $\mathbf{i}_\Theta : \Theta \rightarrow \{1, \dots, |\Theta|\}$ be the one-to-one mapping preserving the order on Θ , i.e., $\mathbf{i}_\Theta(s_1) < \mathbf{i}_\Theta(s_2)$ if and only if $s_1 < s_2$. We also write $\mathbf{i}(s)$ when Θ is clear from the context. The algorithm first substitutes the values of $v_r, r \in S$ into (1) and obtain

$$[u_1, \dots, u_B] - [v_r, r \in S] \mathbf{H}^{\mathbf{i}[S]} = [v_s, s \in \Theta \setminus S] \mathbf{H}^{\mathbf{i}[\Theta \setminus S]}, \quad (4)$$

where $\mathbf{H}^{\mathbf{i}[S]}$ is the submatrix of \mathbf{H} formed by the rows indexed by $\mathbf{i}[S]$. The algorithm then applies Gaussian (Gauss-Jordan) elimination on the above linear system so that $\mathbf{H}^{\mathbf{i}[\Theta \setminus S]}$ is transformed into the reduced column echelon form $\tilde{\mathbf{H}}$ and (4) becomes

$$[\tilde{u}_1, \dots, \tilde{u}_B] = [v_s, s \in \Theta \setminus S] \tilde{\mathbf{H}}. \quad (5)$$

Suppose that the j -th column of $\tilde{\mathbf{H}}$ has only one nonzero component (which should be one) at the row corresponding

to user s . The value of v_s is then \tilde{u}_j and hence recovered. The algorithm returns the new recovered input packets by searching the columns of $\tilde{\mathbf{H}}$ with only one non-zero component.

For a batch with degree d , the complexity of the above decoding is $O(d^3 + d^2T)$. Suppose that K/n is a constant and the maximum degree D does not change with K . Since the degree of a batch converge to the Poisson distribution with parameter $\frac{K}{n} \bar{\Lambda}$ (see Lemma 1), the average complexity of decoding a batch is $O(T)$ finite-field operations. Hence the total decoding complexity is $O(KT)$ finite-field operations.

C. Decoding Performance

For an integer j , denoted by $[j]$ the set of integers $\{1, \dots, j\}$. When $j \leq 0$, $[j] = \emptyset$. For any $\mathbf{H} \in \mathcal{H}_d$, define $\gamma(\mathbf{H})$ as the collection of all subsets V of $[d-1]$ such that in the linear system (1), $v_{\mathbf{i}^{-1}(d)}$ can be uniquely solved when the values of $v_r, r \in \mathbf{i}^{-1}[V]$ are known. Taking the transfer matrix in (3) as an example, we have

$$\gamma(\mathbf{H}) = \{\{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

For a timeslot of degree one, the transfer matrix \mathbf{H} is the one-by-one matrix with the unity. Then $\gamma(\mathbf{H}) = \{\emptyset\}$. For any integer $k \geq 0$, define

$$\Gamma_k(x) = \sum_{\mathbf{H} \in \mathcal{H}_{k+1}} g(\mathbf{H}|k+1) \sum_{S \in \gamma(\mathbf{H})} x^{|S|} (1-x)^{k-|S|}.$$

In other words, $\Gamma_k(x)$ is the probability that when $k+1$ users transmitted in a timeslot, the input packets of the user with the largest index can be recovered if each of the other users' packet is known with probability x .

We assume that the maximum degree D is a constant that does not change with K . The following theorem, proved in the next section, tells us the decoding performance of l iterations of the batched BP decoder when K is sufficiently large. We apply the convention that $0^0 = 1$.

Theorem 2. Fix real numbers $R > 0$, $\epsilon > 0$ and an integer $l > 0$. Consider a multiple-access system described above with K users and $n = \lceil K/R \rceil$ timeslots. Define

$$z_l^* = 1 - \Lambda \left(1 - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \Gamma_k(z_{l-1}) \right),$$

where $z_0 = 0$ and for $1 \leq i < l$

$$z_i = 1 - \Lambda' \left(1 - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \Gamma_k(z_{i-1}) \right) / \bar{\Lambda},$$

where $\lambda = R\bar{\Lambda}$. Then for any sufficiently large K , l iterations of the batch BP decoder will recover at least $K(z_l^* - \epsilon)$ input packets with probability at least $1 - \exp(-c\epsilon^2 K)$, where c is a number independent of K and n .

Proof: See Section III. ■

Lemma 3. $\Gamma_k(x)$ is an increasing function of x .

Proof: This lemma can be proved by applying [21, Lemma 13]. ■

D. Degree Distribution Optimization

Theorem 2 induces a general approach to optimize the degree distribution Λ . Let

$$f(x; \lambda) = 1 - \Lambda' \left(1 - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \Gamma_k(x) \right) / \bar{\Lambda}.$$

We have $z_i = f(z_{i-1}; \lambda)$, $i = 1, \dots, l-1$. Suppose that we allow $l \rightarrow \infty$. The sequence $\{z_i\}$ is increasing (implied by Lemma 3) and converges to the first value $x > 0$ such that $f(x; \lambda) = x$. For given value of λ , $0 < \epsilon < 1$ and $0 < \eta \leq 1$, we can optimize the degree distribution Λ by solving

$$\begin{aligned} \max R \\ \text{s.t. } f(x; \lambda) \geq x(1 + \epsilon), \quad \forall x \in (0, \eta], \\ \sum_i i \Lambda_i = \lambda/R, \sum_i \Lambda_i = 1, \Lambda_i \geq 0. \end{aligned} \quad (6)$$

Theorem 4. Denote by $R(\lambda, \epsilon)$ the optimal value of the above optimization. Then the rate

$$R^*(\lambda, \epsilon) = R(\lambda, \epsilon) \left(1 - \Lambda \left(1 - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \Gamma_k(\eta) \right) \right)$$

packet per timeslot is achievable for the batched BP decoding algorithm.

Proof: For any $\delta > 0$, let $R = R(\lambda, \epsilon) - \sqrt{\delta}$. We show that for sufficiently large K , there exists a degree distribution Λ such that using $n \leq K/R$ timeslots, the batch BP decoding algorithm can recover at least $K(\eta^* - \sqrt{\delta})$ input packets with high probability, where

$$\eta^* = \left(1 - \Lambda \left(1 - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \Gamma_k(\eta) \right) \right).$$

That is the code has a rate at least $R^*(\lambda, \epsilon) - \delta$ packet per timeslot.

Let $n = \lceil K/R(\lambda, \epsilon) \rceil$. For the degree distribution Λ achieving $R(\lambda, \epsilon)$ in (6), we know by Theorem 2 that at $K(z_l^* - \sqrt{\delta})$ input packets can be recovered with high probability. We know that the sequence $\{z_i\}$ converges to a value larger than η . Then there exists a sufficiently large l such that $z_{l-1} \geq \eta$. Thus, $z_l^* \geq \eta^*$. The proof is completed. ■

III. PERFORMANCE ANALYSIS

We generalize the tree-based approach [19] to analyze the performance of the batched BP decoder and prove Theorem 2.

A. Decoding Graph

The relation between the input packets and the timeslots can be represented by a random Tanner graph G , where the input packets are represented by the variable nodes, and timeslots are represented by the check nodes. We henceforth equate a variable node with the corresponding input packet, and a check node with the corresponding timeslot. There exists an edge between a variable node and a check node if and only if the corresponding input packet is transmitted in the timeslot. Associated with each check node is a random transfer matrix

H . For given degree d of the timeslot, the distribution of H is $g(\cdot|d)$.

The l -neighborhood of a variable node v , denoted by $G_l(v)$, is the subgraph of G that includes all the nodes with distance less than or equal to l from variable node v , as well as all the edges involved. Since $G_l(v)$ has the same distribution for all variable node v , we denote by G_l the generic random graph with the same distribution as $G_l(v)$. After l iterations of the batched BP decoding, whether or not a variable node v is decoded is determined by its $2l$ -neighborhood.

Motivated by the tree-based approach, in the remainder of this section, we first analyze the decodable probability of the root node of a random tree, and then show that the decoding performance of G_{2l} is similar to that of the tree. The proof of Theorem 2 is then completed by a martingale argument.

B. Tree Analysis

Fix two degree distributions $\alpha(x)$ and $\beta(x)$. Let T_l be a tree of $l+1$ levels. The root of the tree is at level 0 and the leaves are at level l . Each node at an even level is a variable node, and each node at an odd level is a check node. The probability that the root node has i children is Λ_i . Except for the root node, all the other variable nodes have i children with probability α_i . All the check node has i children with probability β_i . An instance of T_4 is shown in Fig. 1.

Lemma 5. Let x_l^* be the probability that the root variable node is decodable by applying the batched BP decoding on T_{2l} . We have

$$x_l^* = 1 - \Lambda \left(1 - \sum_k \beta_k \Gamma_k(x_{l-1}) \right),$$

where $x_0 = 0$ and for $1 \leq i < l$,

$$x_i = 1 - \alpha \left(1 - \sum_k \beta_k \Gamma_k(x_{i-1}) \right).$$

Proof: Denote by y_i the probability that a check node at level $2(l-i)+1$ can recover its parent variable node by solving the associated linear system of this check node with possibly the knowledge of its children variable nodes. We have $x_l^* = 1 - \Lambda(1 - y_l)$. Suppose that a variable node at level $2(l-i)$ is decodable by at least one of its children check node with probability \hat{x}_i , $0 \leq i < l$. We have $\hat{x}_i = 1 - \alpha(1 - y_i)$ for $0 < i < l$ and $\hat{x}_0 = 0$.

Fix a check node c at level $2(l-i)+1$. With probability $g(\mathbf{H}|k+1)\beta_k$, the check node has k children and the associated linear system has \mathbf{H} as the transfer matrix. We permute the rows of \mathbf{H} such that the *last* row of \mathbf{H} corresponds to the parent variable node. By (2), the permutation does not change the distribution $g(\mathbf{H}|k+1)$. Index the k children by $1, \dots, k$. Using Gaussian elimination in the batched BP decoder, the parent variable node of check node c can be recovered if and only if for certain $S \in \gamma(\mathbf{H})$, all the children variable nodes indices by S are decodable. Therefore, the probability that the parent variable node of c is decodable is $\sum_{S \in \gamma(\mathbf{H})} \hat{x}_{i-1}^{|S|} (1 - \hat{x}_{i-1})^{k-|S|}$ for transfer matrix \mathbf{H} . Considering all the possible transfer matrices, we have $y_i = \sum_k \beta_k \Gamma_k(\hat{x}_{i-1})$. The proof is completed by $x_i = \hat{x}_i$. ■

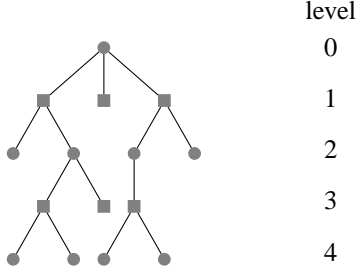


Fig. 1. An instance of T_4 .

We prove the following stronger result than Lemma 1.

Lemma 6. *Suppose that $K/n \rightarrow R$ as $K \rightarrow \infty$. Fix a timeslot t and an integer $k \geq 0$. Under the condition that a fixed set of k users do not transmit at timeslot t , the degree of timeslot t converges to the Poisson distribution with parameter $\lambda = R\bar{\Lambda}$ as $K \rightarrow \infty$.*

Proof: Let Θ be the set of users that do not transmit at timeslot t . For each user that is not in Θ , the probability that this user transmits a packet at timeslot t is $\bar{\Lambda}/n$, when n is larger than D . Therefore, the degree of timeslot t follows a binomial distribution with parameter $(K - k, \bar{\Lambda}/n)$, which converges to the Poisson distribution with parameter $R\bar{\Lambda}$ when $K \rightarrow \infty$. ■

For a positive integer L , let $\epsilon_L = 1 - \sum_{d=0}^L \frac{\lambda^d e^{-\lambda}}{d!}$. We are interested in the following instances of α and β

$$\alpha(x) = \frac{\Lambda'(x)}{\bar{\Lambda}}, \beta(x) = \frac{1}{1 - \epsilon_L} \sum_{k=0}^L \frac{\lambda^k e^{-\lambda}}{k!} x^k. \quad (7)$$

Let $\mathcal{G}_l(L)$ be the set of trees of $l+1$ levels where each check node has at most L children and each variable node has at most D children.

Lemma 7. *When K is sufficiently large, for any $\mathbf{G}_l \in \mathcal{G}_l(L)$,*

$$\Pr\{G_l = \mathbf{G}_l\} \geq \Pr\{T_l = \mathbf{G}_l\} - c_{l,L}\epsilon_L,$$

where $c_{l,L} = O(L^{\lfloor l/2 \rfloor})$ and the degree distributions of T_l are given in (7).

Proof: We show by induction that

$$\Pr\{G_l = \mathbf{G}_l\} \geq \Pr\{T_l = \mathbf{G}_l\} - c_{l,L}\epsilon_L, \quad (8)$$

where $c_{l,L} = O(L^{\lfloor l/2 \rfloor})$.

We prove the lemma by induction. When $l = 1$, G_1 and T_1 follow the same distribution. For $l > 1$, we have

$$\Pr\{G_l = \mathbf{G}_l\} = \Pr\{G_l = \mathbf{G}_l | G_{l-1} = \mathbf{G}_{l-1}\} \Pr\{G_{l-1} = \mathbf{G}_{l-1}\},$$

where \mathbf{G}_{l-1} is the subgraph of \mathbf{G}_l obtained by removing the leaf nodes. We assume that

$$\Pr\{G_{l-1} = \mathbf{G}_{l-1}\} \geq \Pr\{T_{l-1} = \mathbf{G}_{l-1}\} - c_{l-1,L}\epsilon_L,$$

for certain function $c_{l-1,L} = O(L^{\lfloor (l-1)/2 \rfloor})$. We then prove (8) with $l > 0$ for two cases: l is even and l is odd.

We first consider the case that l is even. Suppose that \mathbf{G}_{l-1} has N leaf *check* nodes, which are at level $l-1$ of \mathbf{G}_l . Denote by k_i the number of children variable nodes of the i -th check node at level $l-1$ in \mathbf{G}_l . Since $\mathbf{G}_l \in \mathcal{G}_l(L)$, we have $k_i \leq L$. By Lemma 6, we have

$$\Pr\{G_l = \mathbf{G}_l | G_{l-1} = \mathbf{G}_{l-1}\} \rightarrow \prod_{i=1}^N \frac{\lambda^{k_i} e^{-\lambda}}{k_i!}, \quad K \rightarrow \infty.$$

On the other hand, we have

$$\Pr\{T_l = \mathbf{G}_l | T_{l-1} = \mathbf{G}_{l-1}\} = \frac{1}{(1 - \epsilon_L)^N} \prod_{i=1}^N \frac{\lambda^{k_i} e^{-\lambda}}{k_i!}.$$

Therefore, for sufficiently large K ,

$$\begin{aligned} \Pr\{G_l = \mathbf{G}_l | G_{l-1} = \mathbf{G}_{l-1}\} - \Pr\{T_l = \mathbf{G}_l | T_{l-1} = \mathbf{G}_{l-1}\} \\ \geq (1 - \epsilon_L)^N - 1 - \epsilon_L \\ \geq -(N+1)\epsilon_L. \end{aligned}$$

Note that $N = O(L^{\lfloor l/2 \rfloor})$.

We then consider the case that l is odd. Suppose that \mathbf{G}_{l-1} has N leaf *variable* nodes, which are at level $l-1$ of \mathbf{G}_l . Denote by k_i the number of children check nodes of the i -th variable node at level $l-1$ of \mathbf{G}_l . We know that $k_i \leq D-1$. We then have

$$\begin{aligned} \Pr\{G_l = \mathbf{G}_l | G_{l-1} = \mathbf{G}_{l-1}\} \\ \rightarrow \prod_{i=1}^N \frac{(k_i + 1)\Lambda_{k_i+1}}{\sum_d d\Lambda_d} \\ = \Pr\{T_l = \mathbf{G}_l | T_{l-1} = \mathbf{G}_{l-1}\}. \end{aligned}$$

Therefore, for sufficiently large K , $\Pr\{G_l = \mathbf{G}_l | G_{l-1} = \mathbf{G}_{l-1}\} \geq \Pr\{T_l = \mathbf{G}_l | T_{l-1} = \mathbf{G}_{l-1}\} - \epsilon_L$. ■

C. Proof of Theorem 2

Now we are ready to prove Theorem 2. We say G_l or T_l is decodable if its root is decodable by the batched BP decoding algorithm. Fix a sufficiently large L . We have

$$\begin{aligned} \Pr\{G_{2l} \in \mathcal{G}_{2l}(L) \text{ and is decodable}\} \\ \geq \sum_{\mathbf{G} \in \mathcal{G}_{2l}(L)} \Pr\{\mathbf{G} \text{ is decodable}\} \Pr\{G_{2l} = \mathbf{G}\} \\ \geq \sum_{\mathbf{G} \in \mathcal{G}_{2l}(L)} \Pr\{\mathbf{G} \text{ is decodable}\} \left(\Pr\{T_{2l} = \mathbf{G}\} - \frac{\epsilon}{4|\mathcal{G}_{2l}(L)|} \right) \\ \geq \Pr\{T_{2l} \text{ is decodable}\} - \epsilon/4 = x_l^* - \epsilon/4 \geq z_l^* - \epsilon/2, \end{aligned}$$

where the second inequality follows from Lemma 7 and the last inequality follows that $x_l^* \rightarrow z_l^*$ when $L \rightarrow \infty$.

Let A be the number of variable nodes v with $G_{2l}(v) \in \mathcal{G}_{2l}(L)$ and decodable. We have $\mathbb{E}[A] \geq (z_l^* - \epsilon/2)K$. For $i = 1, \dots, K$, denote $Z_i = G_1(v_i)$. Define $X_i = \mathbb{E}[A | Z_1, \dots, Z_i]$. By definition, X_i is a Doob's martingale with $X_0 = \mathbb{E}[A]$ and $X_K = A$. Since the exposure of a variable node will affect the degrees of a constant number of subgraphs $G_{2l}(v)$ with check node degree $\leq L+1$, we have $|X_i - X_{i-1}| \leq$

c' , a constant does not depend on K . Applying the Azuma-Hoeffding Inequality, we have

$$\Pr\{A \leq \mathbb{E}[A] - \epsilon/2K\} \leq \exp\left(-\frac{\epsilon^2 K}{8c'^2}\right).$$

Hence $\Pr\{A > (z_l^* - \epsilon)K\} > 1 - \exp\left(-\frac{\epsilon^2 K}{8c'^2}\right)$. This completes the proof of Theorem 2.

IV. AN EXAMPLE

In this section, we use an example to illustrate how the proposed NCSA scheme works. Here $q = 2$ and $m = 1$. Fix an integer $N \geq 2$. We consider the PNC scheme that has the following outputs: i) When one user transmits in a timeslot, the packet of the user is decoded; ii) When two to N users transmit in a timeslot, one or two binary linear combinations of the input packets are decoded; and iii) When more than N users transmit in a timeslot, nothing is decoded.

Taking $N = 3$ as an example, when one user transmits in a timeslot, the transfer matrix is $\mathbf{H}_1 = [1]$, and $g(\mathbf{H}_1|1) = 1$. When two users transmit in a timeslot, the possible transfer matrices are

$$\mathbf{H}_{21} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{H}_{22} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \mathbf{H}_{23} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

Since \mathbf{H}_{22} and \mathbf{H}_{23} have the same probability, $g(\mathbf{H}_{21}|2) + 2g(\mathbf{H}_{22}|2) = 1$. Now consider that three users transmit in a timeslot. Define

$$\mathbf{H}_{31} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{H}_{32} = \begin{bmatrix} 1 & \\ 1 & \\ & 1 \end{bmatrix}, \mathbf{H}_{33} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}.$$

The possible transfer matrices are given by the row permutations of \mathbf{H}_{3i} , $i = 1, 2, 3$. Note that for two transfer matrices that are permutation of each other, they have the same probability to occur. Thus we have

$$g(\mathbf{H}_{31}|3) + 3g(\mathbf{H}_{32}|3) + 6g(\mathbf{H}_{33}|3) = 1.$$

We then have

$$\begin{aligned} \Gamma_0(x) &= 1, \quad \Gamma_1(x) = g(\mathbf{H}_{21}|2)x + 2g(\mathbf{H}_{22}|2), \\ \Gamma_2(x) &= g(\mathbf{H}_{31}|3)x^2 + g(\mathbf{H}_{32}|3)(1 + 2x) \\ &\quad + g(\mathbf{H}_{33}|3)(8x - 2x^2). \end{aligned}$$

In general, for a timeslot of d users, we denote by \mathbf{H}_{d1} the single column transfer matrix of all ones. For transfer matrices of two columns, there are three types of rows: $[0, 1]$, $[1, 0]$ and $[1, 1]$. Denote by $\mathbf{H}_{d2}(a)$ a generic transfer matrix with a rows of type $[0, 1]$ and $d - a$ rows of type $[1, 0]$. Here $0 < a \leq \lfloor d/2 \rfloor$. All the row permutations of $\mathbf{H}_{d2}(a)$ are possible transfer matrices.

Denote by $\mathbf{H}_{d3}(a_1, a_2)$ a generic transfer matrix with a_1 rows of type $[0, 1]$, a_2 rows of type $[1, 0]$ and $d - a_1 - a_2$ rows of type $[1, 1]$. Here $a_2 \geq a_1 > 0$ and $a_1 + a_2 < d$.

All the row permutations of $\mathbf{H}_{d3}(a_1, a_2)$ are possible transfer matrices. Thus,

$$\begin{aligned} 1 &= g(\mathbf{H}_{d1}|d) + \sum_{a=1}^{\lfloor d/2 \rfloor} \binom{d}{a} g(\mathbf{H}_{d2}(a)|d) \\ &\quad + \sum_{a_1=1}^{d-2} \sum_{a_2=a_1}^{d-a_1-1} \binom{d}{a_1, a_2} g(\mathbf{H}_{d3}(a_1, a_2)|d). \end{aligned}$$

We can then calculate that

$$\begin{aligned} \Gamma_{d-1}(x) &= g(\mathbf{H}_{d1}|d)x^{d-1} \\ &\quad + \sum_{a=1}^{\lfloor d/2 \rfloor} g(\mathbf{H}_{d2}(a)|d) \left[\binom{d-1}{a-1} x^{a-1} + \binom{d-1}{a} x^{d-a-1} \right] \\ &\quad + \sum_{a_1=1}^{d-2} \sum_{a_2=a_1}^{d-a_1-1} g(\mathbf{H}_{d3}(a_1, a_2)|d) \left[\binom{d-1}{a_1-1, a_2} x^{d-a_2-1} \right. \\ &\quad \left. + \binom{d-1}{a_1, a_2-1} x^{d-a_1-1} \right. \\ &\quad \left. + \binom{d-1}{a_1, a_2} x^{d-a_1-a_2-1} (x^{a_1} + x^{a_2} - x^{a_1+a_2}) \right]. \end{aligned}$$

Given average degree λ of a timeslot, the average number of output packets decoded in a timeslot converges to

$$U(\lambda) = \sum_d \frac{\lambda^d e^{-\lambda}}{d!} \sum_{\mathbf{H} \in \mathcal{H}_d} \text{rk}(\mathbf{H}) g(\mathbf{H}|d),$$

when $K \rightarrow \infty$. The achievable rate of the NCSA system is upper bounded by $U(\lambda)$ packets per timeslot. In the case of this example, the achievable rate bound is given by

$$\begin{aligned} U(\lambda) &= \sum_{d=1}^N \frac{\lambda^d e^{-\lambda}}{d!} \left[g(\mathbf{H}_{d1}|d) + 2 \sum_{a=1}^{\lfloor d/2 \rfloor} \binom{d}{a} g(\mathbf{H}_{d2}(a)|d) \right. \\ &\quad \left. + 2 \sum_{a_1=1}^{d-2} \sum_{a_2=a_1}^{d-a_1-1} \binom{d}{a_1, a_2} g(\mathbf{H}_{d3}(a_1, a_2)|d) \right]. \end{aligned}$$

Note that the upper bound is in general not tight since the packets decoded in different timeslots can be the same.

We solve (6) for the above example with the results in Fig. 2, where we assume the uniform distribution for each possible transfer matrices. We also evaluate the corresponding upper bound $U(\lambda)$ for comparison.

REFERENCES

- [1] L. G. Roberts, "ALOHA packet system with and without slots and capture," *ACM SIGCOMM Computer Communication Review*, vol. 5, no. 2, pp. 28–42, 1975.
- [2] E. Casini, R. De Gaudenzi, and O. R. Herrero, "Contention resolution diversity slotted ALOHA (CRDSA): An enhanced random access scheme for satellite access packet networks," *Wireless Communications, IEEE Transactions on*, vol. 6, no. 4, pp. 1408–1419, 2007.
- [3] G. Liva, "Graph-based analysis and optimization of contention resolution diversity slotted ALOHA," *Communications, IEEE Transactions on*, vol. 59, no. 2, pp. 477–487, 2011.
- [4] E. Paolini, G. Liva, and M. Chiani, "High throughput random access via codes on graphs: Coded slotted ALOHA," in *Proc. IEEE ICC*, 2011.
- [5] C. Stefanovic, P. Popovski, and D. Vukobratovic, "Frameless ALOHA protocol for wireless networks," *Communications Letters, IEEE*, vol. 16, no. 12, pp. 2087–2090, 2012.

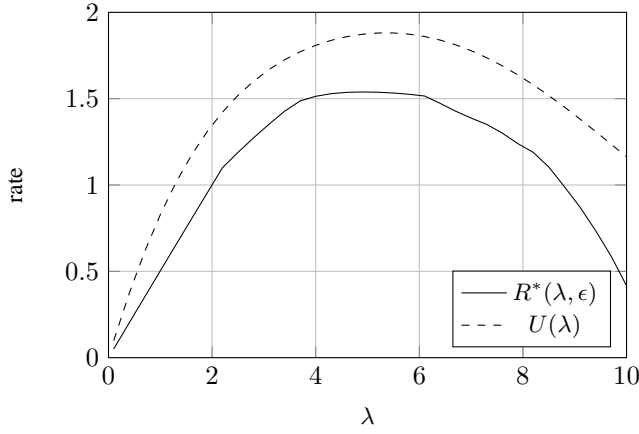


Fig. 2. Achievable rates with $\eta = 0.99$, $N = 10$ and $\epsilon = 0.001$. $R^*(\lambda, \epsilon)$ is the maximum achievable rate optimized w.r.t λ , and $U(\lambda)$ is an upper bound on the maximum achievable rate w.r.t. λ .

- [6] K. R. Narayanan and H. D. Pfister, "Iterative collision resolution for slotted aloha: An optimal uncoordinated transmission policy," in *Proc. IEEE ISTC*, 2012.
- [7] M. Ghanbarinejad and C. Schlegel, "Irregular repetition slotted aloha with multiuser detection," in *Proc. WONS*. IEEE, 2013.
- [8] S. Zhang, S. C. Liew, and P. P. Lam, "Hot topic: Physical-layer network coding," in *Proc. MobiCom '06*, New York, NY, USA, 2006.
- [9] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *Information Theory, IEEE Transactions on*, vol. 57, no. 10, pp. 6463–6486, 2011.
- [10] L. Lu, L. You, and S. C. Liew, "Network-coded multiple access," *Mobile Computing, IEEE Transactions on*, vol. 13, no. 12, pp. 2853–2869, Dec 2014.
- [11] L. You, S. Liew, and L. Lu, "Network-coded multiple access ii : Toward realtime operation with improved performance," *Selected Areas in Communications, IEEE Journal on*, vol. PP, no. 99, pp. 1–1, 2014.
- [12] G. Cocco and S. Pfletschinger, "Seek and decode: Random multiple access with multiuser detection and physical-layer network coding," in *Proc. IEEE ICC*, 2014.
- [13] S. Verdú, *Multiuser detection*. Cambridge university press, 1998.
- [14] S. Zhang and S.-C. Liew, "Channel coding and decoding in a relay system operated with physical-layer network coding," *Selected Areas in Communications, IEEE Journal on*, vol. 27, no. 5, pp. 788–796, June 2009.
- [15] D. Wubben and Y. Lang, "Generalized sum-product algorithm for joint channel decoding and physical-layer network coding in two-way relay systems," in *Proc. IEEE GLOBECOM*, Dec 2010.
- [16] S. C. Liew, S. Zhang, and L. Lu, "Physical-layer network coding: Tutorial, survey, and beyond," (*invited paper*) *Physical Communication*, vol. 6, pp. 4–42, 2013.
- [17] C. Feng, D. Silva, and F. Kschischang, "An algebraic approach to physical-layer network coding," *Information Theory, IEEE Transactions on*, vol. 59, no. 11, pp. 7576–7596, Nov 2013.
- [18] J. Zhu and M. Gastpar, "Gaussian (dirty) multiple access channels: A compute-and-forward perspective," in *Proc. IEEE ISIT*, June 2014.
- [19] M. Luby, M. Mitzenmacher, and M. A. Shokrollahi, "Analysis of Random Processes via And-Or Tree Evaluation," in *Proc. SODA*, 1998.
- [20] S. Yang, S.-C. Liew, L. You, and Y. Chen, "Linearly-coupled fountain codes for network-coded multiple access," in *Proc. IEEE ITW*, Nov 2014.
- [21] —, "Linearly-coupled fountain codes," *arXiv preprint arXiv:1410.2757*, 2014.
- [22] S. Yang and R. Yeung, "Batched sparse codes," *Information Theory, IEEE Transactions on*, vol. 60, no. 9, pp. 5322–5346, Sep 2014.
- [23] S. Yang and B. Tang, "From LDPC to chunked network codes," in *Proc. IEEE ITW*, Nov 2014.